

Newton equation for canonical, Lie-algebraic and quadratic deformation of classical space

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Abstract

The Newton equation describing the particle motion in constant external field force on canonical, Lie-algebraic and quadratic space-time is investigated. We show that for canonical deformation of space-time the dynamical effects are absent, while in the case of Lie-algebraic noncommutativity, when spatial coordinates commute to the time variable, the additional acceleration of particle is generated. We also indicate, that in the case of spatial coordinates commuting in Lie-algebraic way, as well as for quadratic deformation, there appear additional velocity and position-dependent forces.

1 Introduction

Due to several theoretical arguments (see e.g. [1]-[4]) the interest in studying of space-time noncommutativity is growing rapidly. There appeared a lot of papers dealing with noncommutative classical ([5]-[14]) and quantum ([15]-[22]) mechanics, as well as with field theoretical models (see e.g. [23]-[33]), defined on quantum space-time.

At present, in accordance with the Hopf-algebraic classification of all deformations of relativistic and nonrelativistic symmetries [34], [35], one can distinguish three kinds of space-time noncommutativity:

1) Canonical (soft) deformation

$$[x_\mu, x_\nu] = i\theta_{\mu\nu}, \quad (1)$$

with tensor $\theta_{\mu\nu}$ being constant and antisymmetric ($\theta_{\mu\nu} = -\theta_{\nu\mu}$). The explicit form of corresponding Poincare Hopf algebra has been provided in [36], while its nonrelativistic counterpart has been proposed in [37].

2) Lie-algebraic case

$$[x_\mu, x_\nu] = i\theta_{\mu\nu}^\rho x_\rho, \quad (2)$$

with particularly chosen constant coefficients $\theta_{\mu\nu}^\rho$. This kind of space-time modification is represented by κ -Poincare [38], [39] and κ -Galilei [40] Hopf algebras. Besides, the Lie-algebraic twist deformations of relativistic and nonrelativistic symmetries have been provided in [41], [42] and [37]¹.

3) Quadratic deformation

$$[x_\mu, x_\nu] = i\theta_{\mu\nu}^{\rho\tau} x_\rho x_\tau, \quad (3)$$

with constant coefficients $\theta_{\mu\nu}^{\rho\tau}$. Its Hopf-algebraic realization was proposed in [41], [44] in the case of relativistic symmetry, and in [45], for its nonrelativistic counterpart.

In this article we investigate the impact of the mentioned above nonrelativistic deformations (with commuting time direction)² on dynamics of simplest classical system - the nonrelativistic particle moving in a field of constant force. We indicate that in the case of soft deformation the Newton equation is not modified, while for the Lie-algebraic noncommutativity we recover two interesting dynamical effects. First of them corresponds to the case, when commutator of two spatial directions closes to time coordinate, and then, such a kind of noncommutativity additionally produces the acceleration of moving particle. For the second type of Lie-algebraic deformation, when the commutator of spatial directions closes to space coordinates, there are generated velocity and position-dependent forces, i.e. forces, which depend on velocity (\dot{x}) and position (x) of moving particle, respectively³.

¹There also exist so-called fuzzy space noncommutativity [43]. However, in this article such a type of deformation will be not under consideration.

²We consider only spatial deformations, i.e. time plays a role of parameter.

³In the case of "position force" one can recognize well-known inverted oscillator force (see e.g. [46]).

In the case of quadratic deformation the situation appears most complicated. Similarly to the Lie-algebraic case, this type of noncommutativity generates new velocity as well as position-dependent forces, but this time, with an explicit time-dependence. In this paper the analytic form of the corresponding solutions is presented and analyzed in detail.

The paper is organized as follows. In first Section we review some known facts concerning the classical mechanics on canonically deformed quantum space (see e.g [5]). We indicate that in such a case the Newton equation for particle moving in a constant force remains unchanged. In Section 2 we analyze two cases of Lie-algebraic deformations, and we provide the corresponding phase spaces as well as we solve suitable Newton equations. Section 3 deals with the quadratic deformation of classical space. The corresponding Newton equation is provided and its solution is studied as well. The results are summarized and discussed in the last Section.

2 Canonical noncommutativity

Let us start with a set of variables ζ^a with $a = 1, 2, \dots, 2n$. For arbitrary two functions $F(\zeta^a)$ and $G(\zeta^a)$ we define Poisson bracket as follows ([47]; for application to noncommutative space-time see [9])

$$\{F, G\} = \{\zeta^a, \zeta^b\} \frac{\partial F}{\partial \zeta^a} \frac{\partial G}{\partial \zeta^b} . \quad (4)$$

In terms of the above structure and given Hamiltonian $H = H(\zeta^a)$ one can write the equations of motion as

$$\dot{\zeta}^a = \{\zeta^a, H\} . \quad (5)$$

In general case (for any function F depending on ζ^a) we have

$$\dot{F} = \{F, H\} . \quad (6)$$

Below, we will consider the phase space given by $\zeta^a = (x_i, p_i)$ with $i = 1, 2, 3$.

Let us start with canonical type of noncommutativity

$$\{x_i, x_j\} = \theta_{ij} , \quad (7)$$

supplemented by

$$\{p_i, p_j\} = 0 \quad , \quad \{x_i, p_j\} = \delta_{ij} . \quad (8)$$

The relations (7) and (8) define the symplectic structure for the soft deformation of classical (commutative) space, which was studied in [5]-[7].

In accordance with (5) for the Hamiltonian

$$H(\vec{p}, \vec{x}) = \frac{\vec{p}^2}{2m} + V(\vec{x}) , \quad (9)$$

we get the following equations of motion

$$\dot{x}_i = \theta_{ik} \frac{\partial V}{\partial x_k} + \frac{p_i}{m} \quad , \quad \dot{p}_i = -\frac{\partial V}{\partial x_i} . \quad (10)$$

They lead to the corresponding Newton equation (see e.g. [5])

$$m\ddot{x}_i = m\theta_{ik} \frac{d}{dt} \left(\frac{\partial V}{\partial x_k} \right) - \frac{\partial V}{\partial x_i} = m\theta_{ik} \frac{\partial^2 V}{\partial x_k \partial x_l} \dot{x}_l - \frac{\partial V}{\partial x^i} , \quad (11)$$

which for potential⁴

$$V(\vec{x}) = - \sum_{i=1}^3 F_i x_i \quad , \quad \frac{\partial V}{\partial x_k} = -F_k = \text{const.} , \quad (12)$$

remains not deformed

$$m\ddot{x}_i = -\frac{\partial V}{\partial x_i} = F_i . \quad (13)$$

Hence, we see that the canonical space-time deformation (7) does not provide any dynamical effects for particle moving in the potential (12) corresponding to constant force.

3 Lie-algebraic noncommutativity

3.1 Space coordinates commuting to time

Let us consider the Lie-algebraic deformation of space with two spatial directions commuting to time in the following way

$$\{x_i, x_j\} = \frac{1}{\kappa} t (\delta_{i\rho} \delta_{j\tau} - \delta_{i\tau} \delta_{j\rho}) , \quad (14)$$

where κ is the mass-like deformation parameter; indices ρ, τ are different and fixed. As it was already mentioned, such a type of noncommutativity has been recovered in a Hopf algebraic framework in [35], [37] with use of the contraction procedure [48], [49]. Its relativistic counterpart has been proposed in [41].

The commutation relations (14) can be extended (in accordance with Jacobi identity) to the whole phase space as follows

$$\{p_i, p_j\} = 0 \quad , \quad \{x_i, p_j\} = \delta_{ij} . \quad (15)$$

In such a case the Hamilton equations (5) take the form

$$\dot{x}_i = t(\delta_{i\rho} \delta_{k\tau} - \delta_{i\tau} \delta_{k\rho}) \frac{1}{\kappa} \frac{\partial V}{\partial x_k} + \frac{p_i}{m} \quad , \quad \dot{p}_i = -\frac{\partial V}{\partial x_i} , \quad (16)$$

while the corresponding Newton equation looks as follows

$$m\ddot{x}_i = t(\delta_{i\rho} \delta_{k\tau} - \delta_{i\tau} \delta_{k\rho}) \frac{m}{\kappa} \frac{d}{dt} \left(\frac{\partial V}{\partial x_k} \right) + (\delta_{i\rho} \delta_{k\tau} - \delta_{i\tau} \delta_{k\rho}) \frac{m}{\kappa} \frac{\partial V}{\partial x_k} - \frac{\partial V}{\partial x_i} . \quad (17)$$

⁴In the case of quantum space the differential calculus is highly-nontrivial (see e.g. [50], [51]). Fortunately, for such a simple linear function like the potential (12) (there is no products of spatial variables) the result of differentiation is classical.

For the simplest potential (12) we have⁵

$$\begin{cases} m\ddot{x}_i &= F_i \\ m\ddot{x}_\rho &= -\frac{m}{\kappa}F_\tau + F_\rho \\ m\ddot{x}_\tau &= \frac{m}{\kappa}F_\rho + F_\tau, \end{cases} \quad (18)$$

with index i different from ρ and τ .

By trivial integration one can find the following solution of the above system

$$x_i(t) = \frac{F_i}{2m}t^2 + v_{i0}t + x_{i0} \quad , \quad x_\tau(t) = \frac{(\frac{m}{\kappa}F_\rho + F_\tau)}{2m}t^2 + v_{\tau0}t + x_{\tau0} \quad , \quad (19)$$

$$x_\rho(t) = \frac{(-\frac{m}{\kappa}F_\tau + F_\rho)}{2m}t^2 + v_{\rho0}t + x_{\rho0} \quad , \quad (20)$$

where x_{a0} and v_{a0} ($a = k, l$) denote initial positions and velocities, respectively.

We see, that the noncommutativity (14) generates additional acceleration of particle in fixed directions ρ and τ . In direction i the motion of particle remains undeformed. Of course, for $\kappa \rightarrow \infty$, the above solutions become classical and describe particle moving in external constant force $\vec{F} = [F_i, F_\tau, F_\rho]$.

3.2 Space coordinates commuting to space

Let us now turn to the case when two spatial directions commute to the spatial ones⁶

$$\{x_k, x_\gamma\} = \frac{1}{\hat{\kappa}}x_l \quad , \quad \{x_l, x_\gamma\} = -\frac{1}{\hat{\kappa}}x_k \quad , \quad \{x_k, x_l\} = 0 \quad , \quad (21)$$

and where indices k, l, γ are different and fixed. Such a type of noncommutativity has been proposed in the case of nonrelativistic symmetry in [35] as the translation sector of classical Poisson-Lie structure, and in [37], as the Hopf module of quantum Galilei algebra. Its relativistic counterpart has been obtained in [41].

The corresponding phase space is given by the Poisson brackets (21) augmented by

$$\{p_k, x_\gamma\} = \frac{1}{\hat{\kappa}}p_l \quad , \quad \{p_l, x_\gamma\} = -\frac{1}{\hat{\kappa}}p_k \quad , \quad \{x_i, p_j\} = \delta_{ij} \quad ,$$

⁵Due to the fact, that in the equation (18) there is no product of two spatial (noncommutative) positions and velocities, the considering equation is represented on commutative space by the formula (18) as well. In other words, we can pass with Newton equation (18) to the undeformed space without using any star product [37] (a Weyl map [52]). The same situation appears as well in the case of others considered deformations.

⁶ $[\hat{\kappa}] = \text{N} \cdot \text{s}$.

$$\{x_\gamma, p_\gamma\} = 1, \quad \{p_a, p_b\} = 0,$$

where indices i, j are different from γ and $a, b = 1, 2, 3$.

Using the formula (5) one can find the following equations of motion

$$\dot{x}_k = \frac{p_k}{m} + x_l \frac{1}{\hat{\kappa}} \frac{\partial V}{\partial x_\gamma}, \quad \dot{p}_k = p_l \frac{1}{\hat{\kappa}} \frac{\partial V}{\partial x_\gamma} - \frac{\partial V}{\partial x_k}, \quad (22)$$

$$\dot{x}_l = \frac{p_l}{m} - x_k \frac{1}{\hat{\kappa}} \frac{\partial V}{\partial x_\gamma}, \quad \dot{p}_l = -\frac{\partial V}{\partial x_l} - p_k \frac{1}{\hat{\kappa}} \frac{\partial V}{\partial x_\gamma}, \quad (23)$$

in k, l -directions, and

$$\dot{x}_\gamma = \frac{p_\gamma}{m} - x_l \frac{1}{\hat{\kappa}} \frac{\partial V}{\partial x_k} + x_k \frac{1}{\hat{\kappa}} \frac{\partial V}{\partial x_l}, \quad (24)$$

$$\dot{p}_\gamma = -\frac{\partial V}{\partial x_\gamma}, \quad (25)$$

in γ -direction.

The corresponding Newton equations can be find with use of (22)-(25)

$$\begin{aligned} m\ddot{x}_k &= -\frac{\partial V}{\partial x_k} + x_l \frac{m}{\hat{\kappa}} \frac{d}{dt} \left(\frac{\partial V}{\partial x_\gamma} \right) + \dot{x}_l \frac{2m}{\hat{\kappa}} \frac{\partial V}{\partial x_\gamma} + \\ &\quad + mx_k \left(\frac{1}{\hat{\kappa}} \right)^2 \left(\frac{\partial V}{\partial x_\gamma} \right)^2, \end{aligned} \quad (26)$$

$$\begin{aligned} m\ddot{x}_l &= -\frac{\partial V}{\partial x_l} - x_k \frac{m}{\hat{\kappa}} \frac{d}{dt} \left(\frac{\partial V}{\partial x_\gamma} \right) - \dot{x}_k \frac{2m}{\hat{\kappa}} \frac{\partial V}{\partial x_\gamma} + \\ &\quad + mx_l \left(\frac{1}{\hat{\kappa}} \right)^2 \left(\frac{\partial V}{\partial x_\gamma} \right)^2, \end{aligned} \quad (27)$$

$$\begin{aligned} m\ddot{x}_\gamma &= -\frac{\partial V}{\partial x_\gamma} - x_l \frac{m}{\hat{\kappa}} \frac{d}{dt} \left(\frac{\partial V}{\partial x_k} \right) - \dot{x}_l \frac{m}{\hat{\kappa}} \frac{\partial V}{\partial x_k} + \\ &\quad + x_k \frac{m}{\hat{\kappa}} \frac{d}{dt} \left(\frac{\partial V}{\partial x_l} \right) + \dot{x}_k \frac{m}{\hat{\kappa}} \frac{\partial V}{\partial x_l}, \end{aligned} \quad (28)$$

and for the potential (12) they look as follows

$$\begin{cases} m\ddot{x}_k &= F_k - \frac{2m}{\hat{\kappa}} F_\gamma \dot{x}_l + m \left(\frac{F_\gamma}{\hat{\kappa}} \right)^2 x_k \\ m\ddot{x}_l &= F_l + \frac{2m}{\hat{\kappa}} F_\gamma \dot{x}_k + m \left(\frac{F_\gamma}{\hat{\kappa}} \right)^2 x_l \\ m\ddot{x}_\gamma &= F_\gamma + \frac{m}{\hat{\kappa}} F_k \dot{x}_l - \frac{m}{\hat{\kappa}} F_l \dot{x}_k . \end{cases} \quad (29)$$

We see, that this kind of space-time deformation generates velocity ($F \sim \dot{x}$) and position-dependent ($F \sim x$) forces corresponding to both directions k and l . As it was mentioned in Introduction (see footnote 3), in the position dependent force we recognize well-known inverted oscillator force [46]. Besides, by direct calculations one can also check that the solution of above system is given by formulae

$$\begin{aligned} x_k(t) &= -\frac{F_k \hat{\kappa}^2}{F_\gamma^2 m} + \left[t \left(\frac{F_l \hat{\kappa}}{F_\gamma m} + v_{k0} + \frac{F_\gamma x_{l0}}{\hat{\kappa}} \right) + x_{k0} + \frac{F_k \hat{\kappa}^2}{F_\gamma^2 m} \right] \cdot \cos \left(\frac{F_\gamma t}{\hat{\kappa}} \right) + \\ &\quad - \left[t \left(\frac{F_k \hat{\kappa}}{F_\gamma m} - v_{l0} + \frac{F_\gamma x_{k0}}{\hat{\kappa}} \right) - x_{l0} + \frac{F_l \hat{\kappa}^2}{F_\gamma^2 m} \right] \cdot \sin \left(\frac{F_\gamma t}{\hat{\kappa}} \right) , \end{aligned} \quad (30)$$

$$\begin{aligned} x_l(t) &= -\frac{F_l \hat{\kappa}^2}{F_\gamma^2 m} + \left[t \left(-\frac{F_k \hat{\kappa}}{F_\gamma m} + v_{l0} - \frac{F_\gamma x_{k0}}{\hat{\kappa}} \right) + x_{l0} + \frac{F_l \hat{\kappa}^2}{F_\gamma^2 m} \right] \cdot \cos \left(\frac{F_\gamma t}{\hat{\kappa}} \right) + \\ &\quad + \left[t \left(\frac{F_l \hat{\kappa}}{F_\gamma m} + v_{k0} + \frac{F_\gamma x_{l0}}{\hat{\kappa}} \right) + x_{k0} + \frac{F_k \hat{\kappa}^2}{F_\gamma^2 m} \right] \cdot \sin \left(\frac{F_\gamma t}{\hat{\kappa}} \right) , \end{aligned} \quad (31)$$

$$\begin{aligned} x_\gamma(t) &= \frac{F_\gamma}{2m} t^2 + v_{0\gamma} t + x_{0\gamma} + t \frac{1}{\hat{\kappa}} (F_l x_{k0} - F_k x_{l0}) + \\ &\quad + \frac{1}{F_\gamma} \left[\frac{1}{F_\gamma} \left(\frac{2F_k^2 \hat{\kappa}^2}{F_\gamma m} + \frac{2F_l^2 \hat{\kappa}^2}{F_\gamma m} + F_l \hat{\kappa} v_{k0} - F_k \hat{\kappa} v_{l0} \right) + 2F_k x_{k0} + 2F_l x_{l0} \right] + \\ &\quad + \left[t \left(\frac{1}{F_\gamma} \left(\frac{F_k^2 \hat{\kappa}}{F_\gamma m} + \frac{F_l^2 \hat{\kappa}}{F_\gamma m} + F_l v_{k0} - F_k v_{l0} \right) + \frac{F_k x_{k0}}{\hat{\kappa}} + \frac{F_l x_{l0}}{\hat{\kappa}} \right) - \frac{F_k \hat{\kappa} v_{k0}}{F_\gamma^2} + \right. \\ &\quad \quad \left. - \frac{F_l \hat{\kappa} v_{l0}}{F_\gamma^2} + \frac{2}{F_\gamma} (F_l x_{k0} - F_k x_{l0}) \right] \cdot \sin \left(\frac{F_\gamma t}{\hat{\kappa}} \right) + \\ &\quad - \left[t \left(\frac{1}{F_\gamma} (F_k v_{k0} + F_l v_{l0}) - \frac{F_l x_{k0}}{\hat{\kappa}} + \frac{F_k x_{l0}}{\hat{\kappa}} \right) + \frac{2F_k^2 \hat{\kappa}^2}{F_\gamma^3 m} + \right. \\ &\quad \left. + \frac{1}{F_\gamma} \left(\frac{2F_l^2 \hat{\kappa}^2}{F_\gamma^3 m} + \frac{F_l \hat{\kappa} v_{k0}}{F_\gamma} - \frac{F_k \hat{\kappa} v_{l0}}{F_\gamma} + 2(F_k x_{k0} + F_l x_{l0}) \right) \right] \cdot \cos \left(\frac{F_\gamma t}{\hat{\kappa}} \right) , \end{aligned} \quad (32)$$

where x_{a0} and v_{a0} ($a = k, l$) denote initial positions and velocities, respectively. The corresponding trajectories are illustrated on Figure 1 for different values of parameter

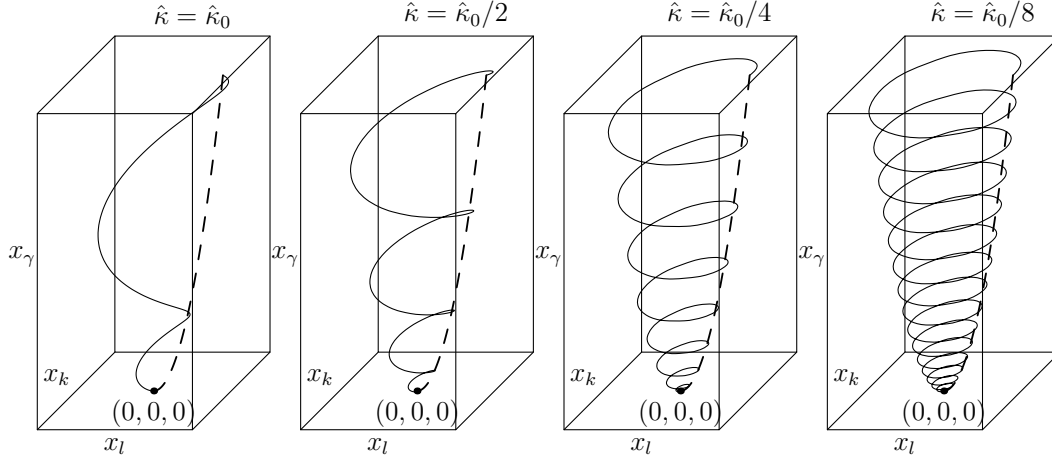


Figure 1: The illustration of particle trajectories for different values of parameter $\hat{\kappa}$ ($\hat{\kappa}_0 = 30$) with nonzero value of m , F_γ and v_{l0} only. The dashed line corresponds to undeformed case ($\hat{\kappa} = \infty$), and the time parameter runs from 0 to $\frac{4\pi\hat{\kappa}_0}{F_\gamma}$.

$\hat{\kappa}$. Their shape indicates that particle moves in γ -direction along vortex line with period $T = \frac{2\pi\hat{\kappa}}{F_\gamma}$. Using the solutions (30)-(32) one can also check that distance between two neighboring rolls of vortex in (k,l) -plane is given by

$$\Delta r = 2\pi \sqrt{\left(-x_{k0} + \frac{v_{l0}\hat{\kappa}}{F_\gamma} - \frac{F_k\hat{\kappa}^2}{F_\gamma^2 m}\right)^2 + \left(x_{l0} + \frac{v_{k0}\hat{\kappa}}{F_\gamma} + \frac{F_l\hat{\kappa}^2}{F_\gamma^2 m}\right)^2}. \quad (33)$$

Besides, it should be noted that for both components F_k , F_l and all initial constants equal zero, the distance Δr vanishes, i.e. particle moves along straight line in γ -direction with constant acceleration $a_\gamma = \frac{F_\gamma}{2m}$. Of course, for parameter $\hat{\kappa} = \infty$, the solutions (30)-(32) become undeformed and describe the motion of classical particle in constant force \vec{F} .

4 Quadratic noncommutativity

Let us now consider the most complicated type of noncommutativity, i.e. the quadratic deformation of classical space [35] (see also [45])

$$\{x_k, x_\gamma\} = \frac{1}{\bar{\kappa}} t x_l, \quad \{x_l, x_\gamma\} = -\frac{1}{\bar{\kappa}} t x_k, \quad \{x_k, x_l\} = 0, \quad (34)$$

with dimensionfull parameter $\bar{\kappa}$ ($[\bar{\kappa}] = \text{N} \cdot \text{s}^2$); indices k, l, γ are different and fixed. Its relativistic counterpart has been proposed in [41] and [44].

The remaining phase space relations are given by

$$\{p_k, x_\gamma\} = \frac{1}{\bar{\kappa}} t p_l, \quad \{p_l, x_\gamma\} = -\frac{1}{\bar{\kappa}} t p_k, \quad \{x_i, p_j\} = \delta_{ij},$$

$$\{x_\gamma, p_\gamma\} = 1, \quad \{p_a, p_b\} = 0,$$

with $i, j \neq \gamma$ and $a, b = 1, 2, 3$. They satisfy the Jacobi identity together with (34). One can check that corresponding equations of motion take the form

$$\dot{x}_k = \frac{p_k}{m} + tx_l \frac{1}{\bar{\kappa}} \frac{\partial V}{\partial x_\gamma}, \quad \dot{p}_k = tp_l \frac{1}{\bar{\kappa}} \frac{\partial V}{\partial x_\gamma} - \frac{\partial V}{\partial x_k}, \quad (35)$$

$$\dot{x}_l = \frac{p_l}{m} - tx_k \frac{1}{\bar{\kappa}} \frac{\partial V}{\partial x_\gamma}, \quad \dot{p}_l = -\frac{\partial V}{\partial x_l} - tp_k \frac{1}{\bar{\kappa}} \frac{\partial V}{\partial x_\gamma}, \quad (36)$$

in k, l -directions, and

$$\dot{x}_\gamma = \frac{p_\gamma}{m} - tx_l \frac{1}{\bar{\kappa}} \frac{\partial V}{\partial x_k} + tx_k \frac{1}{\bar{\kappa}} \frac{\partial V}{\partial x_l}, \quad \dot{p}_\gamma = -\frac{\partial V}{\partial x_\gamma}, \quad (37)$$

in γ -direction.

By direct calculation one can also find the corresponding Newton equations, which look as follows

$$m\ddot{x}_k = -\frac{\partial V}{\partial x_k} + tx_l \frac{m}{\bar{\kappa}} \frac{d}{dt} \left(\frac{\partial V}{\partial x_\gamma} \right) + (2t\dot{x}_l + x_l) \frac{m}{\bar{\kappa}} \frac{\partial V}{\partial x_\gamma} + t^2 x_k \frac{1}{m\bar{\kappa}^2} \left(\frac{\partial V}{\partial x_\gamma} \right)^2, \quad (38)$$

$$m\ddot{x}_l = -\frac{\partial V}{\partial x_l} - tx_k \frac{m}{\bar{\kappa}} \frac{d}{dt} \left(\frac{\partial V}{\partial x_\gamma} \right) - (2t\dot{x}_k + x_k) \frac{m}{\bar{\kappa}} \frac{\partial V}{\partial x_\gamma} + t^2 x_l \frac{1}{m\bar{\kappa}^2} \left(\frac{\partial V}{\partial x_\gamma} \right)^2, \quad (39)$$

$$m\ddot{x}_\gamma = -\frac{\partial V}{\partial x_\gamma} - tx_l \frac{m}{\bar{\kappa}} \frac{d}{dt} \left(\frac{\partial V}{\partial x_k} \right) - (t\dot{x}_l + x_l) \frac{m}{\bar{\kappa}} \frac{\partial V}{\partial x_k} + tx_k \frac{m}{\bar{\kappa}} \frac{d}{dt} \left(\frac{\partial V}{\partial x_l} \right) + (t\dot{x}_k + x_k) \frac{m}{\bar{\kappa}} \frac{\partial V}{\partial x_l}. \quad (40)$$

Obviously, for the potential (12) the set of equations of motion takes the form

$$\begin{cases} m\ddot{x}_k = F_k - \frac{m}{\bar{\kappa}} F_\gamma \left(x_l - \frac{1}{\bar{\kappa}} F_\gamma t^2 x_k \right) - \frac{2m}{\bar{\kappa}} F_\gamma t\dot{x}_l \\ m\ddot{x}_l = F_l + \frac{m}{\bar{\kappa}} F_\gamma \left(x_k + \frac{1}{\bar{\kappa}} F_\gamma t^2 x_l \right) + \frac{2m}{\bar{\kappa}} F_\gamma t\dot{x}_k \\ m\ddot{x}_\gamma = F_\gamma + \frac{m}{\bar{\kappa}} F_k (t\dot{x}_l + x_l) - \frac{m}{\bar{\kappa}} F_l (t\dot{x}_k + x_k) \end{cases} \quad (41)$$

We see that as in the Lie-algebraic case, the quadratic noncommutativity generates the velocity and position-dependent forces, but this time, with time dependent coefficients linear and quadratic in time t .

By direct calculation we get the solutions

$$x_k(t) = A_k(t) \cos\left(\frac{F_\gamma t^2}{2\bar{\kappa}}\right) - A_l(t) \sin\left(\frac{F_\gamma t^2}{2\bar{\kappa}}\right), \quad (42)$$

$$x_l(t) = A_k(t) \sin\left(\frac{F_\gamma t^2}{2\bar{\kappa}}\right) + A_l(t) \cos\left(\frac{F_\gamma t^2}{2\bar{\kappa}}\right), \quad (43)$$

$$x_\gamma(t) = \frac{F_\gamma}{2m} t^2 + v_{\gamma 0} t + x_{\gamma 0} + \frac{1}{\bar{\kappa}} \int_0^t (zx_l(z)F_k - zx_k(z)F_l) dz, \quad (44)$$

where the coefficients $A_k(t)$ and $A_l(t)$ are given by

$$\begin{aligned} A_k(t) = & x_{k0} + v_{k0}t + \frac{F_k \bar{\kappa}}{mF_\gamma} \left[\pi t \sqrt{\frac{F_\gamma}{\pi \bar{\kappa}}} C\left(t \sqrt{\frac{F_\gamma}{\pi \bar{\kappa}}}\right) - \sin\left(\frac{F_\gamma t^2}{2\bar{\kappa}}\right) \right] + \\ & + \frac{F_l \bar{\kappa}}{mF_\gamma} \left[\cos\left(\frac{F_\gamma t^2}{2\bar{\kappa}}\right) - 1 + \pi t \sqrt{\frac{F_\gamma}{\pi \bar{\kappa}}} S\left(t \sqrt{\frac{F_\gamma}{\pi \bar{\kappa}}}\right) \right], \end{aligned} \quad (45)$$

$$\begin{aligned} A_l(t) = & x_{l0} + v_{l0}t + \frac{F_l \bar{\kappa}}{mF_\gamma} \left[\pi t \sqrt{\frac{F_\gamma}{\pi \bar{\kappa}}} C\left(t \sqrt{\frac{F_\gamma}{\pi \bar{\kappa}}}\right) - \sin\left(\frac{F_\gamma t^2}{2\bar{\kappa}}\right) \right] + \\ & - \frac{F_k \bar{\kappa}}{mF_\gamma} \left[\cos\left(\frac{F_\gamma t^2}{2\bar{\kappa}}\right) - 1 + \pi t \sqrt{\frac{F_\gamma}{\pi \bar{\kappa}}} S\left(t \sqrt{\frac{F_\gamma}{\pi \bar{\kappa}}}\right) \right], \end{aligned} \quad (46)$$

and the functions $C(z)$, $S(z)$ are defined as follows

$$C(z) = \int_0^z \cos\left(\frac{\pi t^2}{2}\right) dt, \quad S(z) = \int_0^z \sin\left(\frac{\pi t^2}{2}\right) dt. \quad (47)$$

The corresponding trajectories for different values of parameter $\bar{\kappa}$ are illustrated on Figure 2. Of course, for deformation parameter approaching infinity, the above solution becomes undeformed and describes the classical particle in a field of constant force.

5 Final remarks

In this article we investigate properties of simple classical system in the presence of three known noncommutative manifolds: canonical, Lie-algebraic and quadratic space-times. We indicate that there are no dynamical effects for soft type of deformation, while for the Lie-algebraic and quadratic noncommutativities there appear additional velocity and

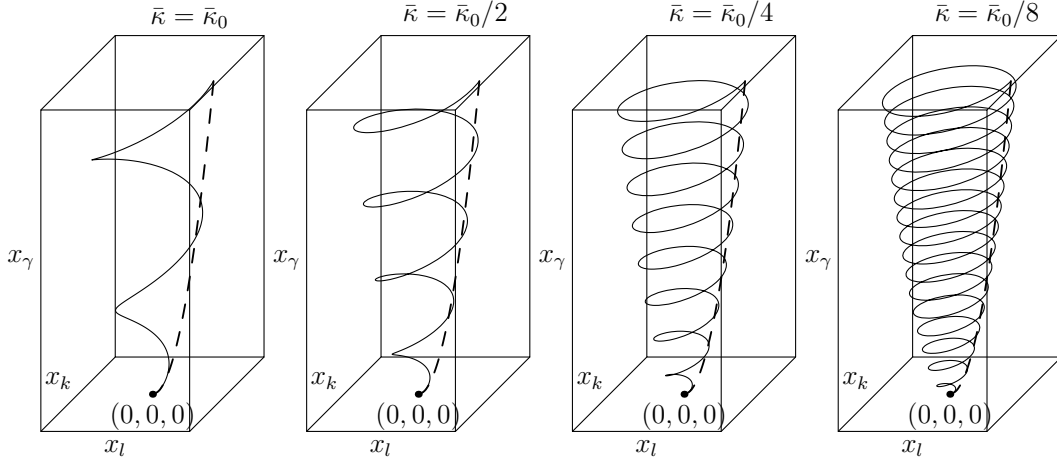


Figure 2: The illustration of particle trajectories for different values of parameter $\bar{\kappa}$ ($\bar{\kappa}_0 = 30$) with nonzero value of m , F_γ and v_{l0} only. The dashed line corresponds to undeformed case ($\bar{\kappa} = \infty$), and the time parameter runs from 0 to $2\sqrt{\frac{2\pi\bar{\kappa}_0}{F_\gamma}}$.

position-dependent forces. The solutions of corresponding Newton equations are provided and analyzed in this paper.

The present studies can be extended in various way. First of all, one can consider more complicated system like the particle in a presence of well-known harmonic oscillator potential - see e.g. [5]. Unfortunately, due to the complicated form of the Newton equations (26)-(28) and (38)-(40) such question appears highly nontrivial and is postponed for subsequent investigations mainly. It is also interesting to consider basic quantum systems in noncommutative space as for example the particle in a hole potential, and find their spectra in the presence of all considered deformations (see e.g. [22]). Finally, one can extend the presented studies to the case of deformed relativistic particle at the classical and quantum level (see [6], [9], [11]). The investigations in these directions already started and are in progress.

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